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Hence by quadrature

$$F = \int_0^{y'} (y' - t) \frac{y}{x(y - xt)} \Phi\left(\frac{-t}{x(y - xt)}, \frac{1}{y(y - xt)}\right) dt + y'\lambda(x, y) + \mu(x, y).$$

It is easily proved that the functions λ and μ must again satisfy the condition

$$\frac{\partial \lambda}{\partial x} = \frac{\partial \mu}{\partial y}.$$

Therefore the most general solution for F is seen to be

$$F(x, y, y') = \int_0^{y'} (y' - t) \frac{y}{x(y - xt)} \Phi\left(\frac{-t}{x(y - xt)}, \frac{1}{y(y - xt)}\right) dt + y'\frac{\partial V}{\partial y} + \frac{\partial V}{\partial x}.$$

Furthermore it can be proved that in neither of the last two cases can the function Φ be so chosen that the transversals are perpendicular to the extremals.

THE PROBABILITY INTEGRAL DEDUCED BY MEANS OF DEVELOPMENTS IN FINITE FORM.

By EDWARD L. DODD, University of Texas.

In modern mathematics, expansions in *finite* form are constantly coming into greater favor in preference to infinite series from which the terms after the first two or three have been dropped.

The object of this paper is to express in modern form the deduction of the probability integral as given by J. Bertrand,[1] whose method follows Laplace's,[2]—though his integral is not quite the same.

Let p be the probability that an event will happen in a single trial, where $0 < p < 1$; and let $q = 1 - p$. Then the probability, P_r, that the event will happen exactly r times on s trials is

$$P_r = \frac{s!}{r!\,(s - r)!} p^r q^{s-r}. \tag{1}$$

Here r is a positive integer not greater than s, or r is zero, accepting unity as the value of $0!$

Now let

$$l = sp - r. \tag{2}$$

Then $r = sp - l$, and

$$P_r = \frac{s!}{(sp - l)!(sq + l)!} p^{sp-l} q^{sq+l}. \tag{3}$$

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[1] *Calcul des Probabilités* (1889), p. 76.

[2] *Œuvres Complètes*, VII, Theorie Analytique des Probabilités, p. 284.

Stirling's[1] Formula, in *finite* form is

$$n! = n^n e^{-n} \sqrt{2\pi n}\, e^{\theta/12n}, \quad 0 < \theta < 1. \tag{4}$$

Using (4), and taking r as a positive integer *less* than s, (3) becomes

$$P_r = \frac{1}{\sqrt{2\pi spq}} \frac{1}{\left(1 - \frac{l}{sp}\right)^{sp-l+\frac{1}{2}}} \frac{1}{\left(1 + \frac{l}{sq}\right)^{sq+l+\frac{1}{2}}} e^{\frac{\theta}{12s} - \frac{\theta'}{12(sp-l)} - \frac{\theta''}{12(sq+l)}}. \tag{5}$$

Now let any positive number c be chosen; and then let s be taken large enough so that

$$sp - c\sqrt{s} > 0, \quad \text{and} \quad sp + c\sqrt{s} < s. \tag{6}$$

And let r be such that

$$sp - c\sqrt{s} \leqq r \leqq sp + c\sqrt{s}. \tag{7}$$

Then

$$|\, l \,| \leqq c\sqrt{s}. \tag{8}$$

For brevity, set

$$u = \left(1 - \frac{l}{sp}\right)^{sp-l+\frac{1}{2}}, \quad v = \left(1 + \frac{l}{sq}\right)^{sq+l+\frac{1}{2}}.$$

Now, if $|\, t \,| < \frac{1}{2}$,

$$\log_e (1 + t) = t - \frac{t^2}{2} + \theta_1 t^3, \quad 0 < \theta_1 < 1. \tag{9}$$

But, if s is sufficiently great, l/sp and l/sq will be numerically less than $\frac{1}{2}$, because of (8). Then, by the use of (9), it may be shown that

$$\log_e uv = \frac{l^2}{2spq} + R, \qquad |\, R \,| < \frac{k}{\sqrt{s}},$$

in which k is some constant; that is, k is independent of s. Indeed, to get an upper bound for R, let all its terms be made positive, let the θ_1 in (9) be taken as unity, and let l be replaced by $c\sqrt{s}$, as in (8). In the expression thus obtained, the numerators are functions of c, and each denominator contains the factor $\sqrt{s}$.

Then,

$$uv = e^{\frac{l^2}{2spq}} \cdot e^{\frac{\theta_2 k}{\sqrt{s}}}, \quad -1 < \theta_2 < 1.$$

Now if $|\, t \,| < \frac{1}{2}$,

$$e^t = 1 + 2\eta t, \quad 0 < \eta < 1. \tag{10}$$

Hence, when s is large enough,

$$e^{-\frac{\theta_2 k}{\sqrt{s}}} = 1 + \frac{2\eta' k}{\sqrt{s}}, \quad -1 < \eta' < 1.$$

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[1] See Cesaro, *Corso di analisi algebrica*, Turin, 1884, pages 270 and 480; or Broggi, *Traité des Assurances sur la Vie*, Trad. S. Lattès, 1907, p. 54.

The relation (10) can also be applied to the last factor in (5), noticing that in $sp - l$, the l becomes insignificant in comparison with sp when s is large, because of (8). Hence

$$P_r = \frac{1}{\sqrt{2\pi spq}} e^{-\frac{l^2}{2spq}} + \epsilon_1, \quad |\epsilon_1| < \frac{K_1}{s}, \tag{11}$$

in which K_1 is some constant; that is, K_1 is independent of s.

For brevity, set

$$h = \frac{1}{\sqrt{2spq}}. \tag{12}$$

Let r_1 and r_2 be any two values of r which satisfy (7), with $r_1 > r_2$; and let l_1 and l_2 correspond by (2). Then,

$$\sum_{r_2}^{r_1} P_r = \frac{h}{\sqrt{\pi}} \sum_{l_1}^{l_2} e^{-h^2 l^2} + \epsilon_2, \quad |\epsilon_2| < \frac{K_2}{\sqrt{s}}, \tag{13}$$

in which K_2 is some constant.

By dropping ϵ_1 in (11), we get an approximation, P_r', for P_r; viz.,

$$P_r' = \frac{h}{\sqrt{\pi}} e^{-h^2 l^2}. \tag{14}$$

Since r is an integer, any two consecutive values of l differ from each other by unity, because of (2). Use the values of l as abscissas and the corresponding values of P_r' as ordinates. With each of these ordinates as a central line, construct a rectangle with its base upon the x-axis, the base being unity in length. The *probability curve*, whose equation is

$$y = f(x) = \frac{h}{\sqrt{\pi}} e^{-h^2 x^2}, \tag{15}$$

passes through the middle points of the upper bases of these rectangles. The probability curve has but two points of inflection, and near the origin is concave towards the x-axis. For this portion of the curve, the following figure may be drawn. See page 126.

In this figure, A is the point on the X axis, whose abscissa is l; AB the ordinate whose length is P_r'; $CDFE$ the rectangle—bisected by AB—with its area numerically equal to P_r'; GBH an arc of the probability curve (15); GH a chord; IBJ the tangent to GBH at B; and K the intersection of AB and GH.

It is now to be proved that if l satisfies (8),

$$P_r' = \frac{h}{\sqrt{\pi}} \int_{l-1/2}^{l+1/2} e^{-h^2 x^2} dx + \epsilon_3, \quad |\epsilon_3| < \frac{K_3}{s\sqrt{s}}, \tag{16}$$

in which K_3 is some constant.

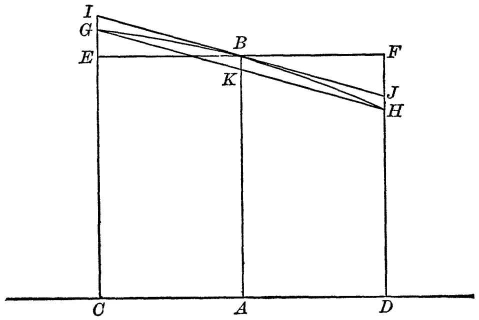
Proof.—In the figure, the trapezoid $CDJI$ has an area numerically equal to P_r'; and this differs from the area beneath the arc GBH by less than the area of the trapezoid $GHJI$. This last mentioned area is $\overline{KB} \times \overline{CD} = \overline{KB}$; since $\overline{CD}$ is of unit length. Hence $\overline{KB}$ is numerically greater than ϵ_3 in (16). But

$$\overline{KB} = f(l) - \tfrac{1}{2}[f(l - \tfrac{1}{2}) + f(l + \tfrac{1}{2})]$$
$$= \frac{he^{-h^2l^2}}{2\sqrt{\pi}}\left[2 - e^{-\frac{h^2}{4}}(e^{h^2l} + e^{-h^2l})\right]. \quad (17)$$

Moreover, if $| t | < \frac{1}{2}$,

$$e^t = 1 + t + \zeta t^2, \qquad 0 < \zeta < 1. \quad (18)$$

And, from (8) and (12), it is evident that $| h^2l |$ becomes less than $\frac{1}{2}$ when s is large enough. Using (18) upon the parenthesis in (17) and also upon $e^{-h^2/4}$, and noticing that $e^{-h^2l^2} \leqq 1$, we prove (16). Beyond the points of inflection the curve (15) is convex towards the X axis; but here, also, a similar treatment applies.



Then, from (11), (14), and (16), we can conclude that

$$\sum_{r_2}^{r_1} P_r = \frac{h}{\sqrt{\pi}} \int_{l_1-\frac{1}{2}}^{l_2+\frac{1}{2}} e^{-h^2x^2}dx + \epsilon_4, \quad | \epsilon_4 | < \frac{K_4}{\sqrt{s}}, \quad (19)$$

in which K_4 is some constant. The discussion has been carried out under the restriction imposed by (8); viz.,

$$- c\sqrt{s} \leqq l_1 < l_2 \leq c\sqrt{s};$$

but the effect of this restriction may be removed by choosing c large enough.

Indeed, let $\alpha = l_2 + \frac{1}{2}$, suppose this positive and nearly equal to $c\sqrt{s}$; and, for convenience, suppose that we can take $l_1 - \frac{1}{2} = -\alpha$. Then the integral in (19) becomes $\Theta(h\alpha)$. Now, by (12), $\alpha < c\sqrt{s}$, provided

$$h\alpha < \frac{c}{\sqrt{2pq}}.$$

But, by taking c large enough, we may take as large values of $h\alpha$ as we wish, making $\Theta(h\alpha)$ differ from unity by as little as we please. For example, if we take $c = 5\sqrt{2pq}$, we may take s and α so that $4.9 < h\alpha < 5$; and $\Theta(h\alpha)$ will differ then from unity by less than 10^{-11}. Hence, by properly choosing c and then s, and taking r_2 and r_1 nearly equal to $sp - c\sqrt{s}$ and $sp + c\sqrt{s}$, respectively, $\sum_{r_2}^{r_1} P_r$ in (19) may be made as near unity as we please. But, by (1), $\sum_0^s P_r = (p + q)^s = 1$. Hence, the rectangles which the restriction (8) excludes are negligible when s is large enough.

By reference to (11), it is evident that no essential change in (19) is made if $l_1 - \frac{1}{2}$ and $l_2 + \frac{1}{2}$ are replaced by l_1 and l_2. respectively

THEOREM: *Let p be the probability that an event will occur in a single trial, where $0 < p < 1$; let s be the number of trials to be made; and let $h = [2sp(1-p)]^{-\frac{1}{2}}$. Then the probability that the number of occurrences of the event will be some number in the set, $sp - l_2$ to $sp - l_1$ inclusive, where $0 \leqq sp - l_2 \leqq sp - l_1 \leqq s$, is*

$$\frac{h}{\sqrt{\pi}} \int_{l_1}^{l_2} e^{-h^2x^2} dx + \delta,$$

where $\lim_{s=\infty} \delta = 0$ *uniformly.*

While this commonly accepted theorem is the most important result of the foregoing set of inequalities, the relations, (11), (13), (16), and (19), are not without some individual importance.

WESTERN MEETINGS OF MATHEMATICIANS

The American Mathematical Society has been for nearly a quarter of a century the official organization through which scientific activity in mathematics has found expression in this country. In 1894 The New York Mathematical Society, which had then been in existence for six years, was expanded into the American Mathematical Society, and the *Bulletin,* which had been published for three years by the former, was continued by the latter as "A Historical and Critical Review of Mathematical Science."

The regular meetings of the Society were held in New York, except in the case of the summer meetings, which, from the first in 1894 to the nineteenth in 1912, were held respectively in Brooklyn, N. Y., Springfield, Mass., Buffalo, N. Y., Toronto, Can., Boston, Mass., Columbus, O., New York, N. Y., Ithaca, N. Y., Evanston, Ill., Boston, Mass., St. Louis, Mo., Williamstown, Mass., New Haven, Conn., Ithaca, N. Y., Urbana, Ill., Princeton, N. J., New York, N. Y., Poughkeepsie, N. Y., and Philadelphia, Pa. It thus appears that of the nineteen summer meetings, four have been held in the middle west, namely, at Columbus, in 1899, at Evanston in 1902, at St. Louis in 1904, and at Urbana in 1908.

Another notable gathering of mathematicians took place at Chicago in 1893, namely, the International Mathematical Congress held in connection with the World's Columbian Exposition. The papers read at this meeting, including nine by Americans and thirty by representatives of other countries, were published for the Society by the Macmillan Company in 1896. There was also held at Evanston, Ill., from August 28 to September 9, 1893, a Colloquium in connection with the World's Fair, consisting of a series of lectures delivered by Professor Felix Klein, of Göttingen, Germany. These lectures have been published by the Society. Other colloquia have been held from time to time in connection with the summer meetings, at which series of lectures have been delivered by members of the Society, but none of these have as yet been held in the west. The next one, however, is to be at Madison, Wis., in September, 1913, in connection with the twentieth summer meeting of the Society.